Contravariant vector:

Tensors are defined using the properties of their transformation rules under coordinate transformations. Vectors are the special case of tensors.

Consider a physial enity is characterized by N functions A^i when expressed in the x^i coordinate system. Let the same enity be characterized by $\bar{A}^\alpha$ when it is measured in coordinate system $\bar{x}^\alpha$.

A^i are called components of a <u>contravariant vector</u> if they transform under coordinates transformations given as

$$\boxed{\bar{A}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j}, \quad \text{--- (1)}$$

which can be inverted to obtain A^i in terms of $\bar{A}^\alpha$ multiplying (1) by $\frac{\partial x^k}{\partial \bar{x}^\alpha}$ and summing over all α.

$$\frac{\partial x^k}{\partial \bar{x}^\alpha} \bar{A}^\alpha = \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j$$

$$\frac{\partial x^k}{\partial \bar{x}^\alpha} \bar{A}^\alpha = \delta_j^k A^j = A^k$$

or $$\boxed{A^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} \bar{A}^\alpha} \quad \text{--- (2)}$$

Covariant vector:

A set of N quantities A_i which are functions of the N coordinates x^i are said to be the components of a covariant vector of they transform given as

$$\bar{A}_\alpha = \frac{\partial x^j}{\partial \bar{x}^\alpha} A_j \quad \text{——} \quad (3)$$

under a change of coordinates from x^i to $\bar{x}^\alpha$, where $\bar{A}_\alpha$ are the components of the vector in the barred coordinate system. Inverse transformation is given by

$$A_i = \frac{\partial \bar{x}^\alpha}{\partial x^i} \bar{A}_\alpha \quad \text{——} \quad (4)$$

Q: Show that velocity and acceleration are contravariant vectors and that the gradient of a scalar field is a covariant vector.

Soln.

We have already obtain relations (see earlier class notes on tensor) given by

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i \quad \text{——} \quad (5)$$

$$\text{and} \quad dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha \quad \text{——} \quad (6)$$

(i) Let t denotes time. Dividing Eq (8) by dt

$$\frac{\partial \bar{x}^\alpha}{dt} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{dx^i}{dt} \quad — (7)$$

Next, we define velocity components in barred and unbarred coordinat systems given by

$$\bar{v}^\alpha = \frac{d\bar{x}^\alpha}{dt}, \quad v^i = \frac{dx^i}{dt}.$$

Now Eq. (7) can be written as

$$\boxed{\bar{v}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} v^i} \quad — (8)$$

See Eq. (1) and compare with (8). v^i is a contravariant vector. If you take derivative of (8) again we get,

$$\frac{d\bar{v}^\alpha}{dt} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{dv^i}{dt}$$

or $$\boxed{\bar{a}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} a^i} \quad — (9) \quad \Big\{ \; a \rightarrow \text{acceleration}$$

acceleration is also a contravariant vector.

(ii) Now consider $\phi \equiv \phi(x^i)$ be a scalar field. Thus, its ~~[illegible]~~ functions form will remain same under coordinate transformation

$$\phi(x^i) = \bar{\phi}(\bar{x}^\alpha) = \phi(\bar{x}^\alpha)$$

Gradient of the scalar field will be a vector whose components can be defined by

$$A_i = \frac{\partial \phi}{\partial x^i}, \quad \bar{A}_\alpha = \frac{\partial \bar{\phi}}{\partial \bar{x}^\alpha} = \frac{\partial \phi}{\partial \bar{x}^\alpha}$$

Next, using partial derivative, we can write

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^i}$$

or $$A_i = \frac{\partial \bar{x}^\alpha}{\partial x^i} \bar{A}_\alpha \quad \text{———} \quad (10)$$

~~See~~ Eq. (4). and Eq. (10), It is clear that gradient of a scalar field is a covariant vector.